An application of decomposable maps in proving multiplicativity of low dimensional maps

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Abstract

In this paper we present a class of maps for which the multiplicativity of the maximal output p-norm holds for $p = 2$ and $p \geq 4$. This result is a slight generalization of the corresponding result in [9]. The class includes all positive trace-preserving maps from $\mathcal{B}(\mathbb{C}^3)$ to $\mathcal{B}(\mathbb{C}^2)$. Interestingly, by contrast, the multiplicativity of p-norm was investigated in the context of quantum information theory and shown not to hold in general for high dimensional quantum channels [5]. Moreover, the Werner-Holevo channel, which is a map from $\mathcal{B}(\mathbb{C}^3)$ to $\mathcal{B}(\mathbb{C}^3)$, is a counterexample for $p > 4.79$.

1 Introduction

Suppose we have a map

$$\Phi : \mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^n), \tag{1.1}$$

where $\mathcal{B}(\mathbb{C}^d)$ is the set of (bounded) linear operators on $\mathbb{C}^d$. Then, the maximal output p-norm is defined as

$$\nu_p(\Phi) = \sup_{\rho \in \mathcal{D}(\mathbb{C}^m)} \|\Phi(\rho)\|_p. \tag{1.2}$$

Here, $\mathcal{D}(\mathbb{C}^m)$ is the set of positive semidefinite Hermitian operators of unit trace, and $\| \ \|_p$ is the Schatten p-norm: $\|A\|_p = (\mathrm{tr}|A|^p)^{\frac{1}{p}}$.

The multiplicativity property was investigated in the context of quantum information theory. I.e., $\mathcal{D}(\mathbb{C}^m)$ represents quantum states on the m-dimensional space, and we restrict the map Φ in (1.2) to Completely Positive (CP) Trace-Preserving (TP) maps, which represent quantum channels. Recall that a map Φ is CP if for any space $\mathbb{C}^d$ the product $\Phi \otimes 1_{\mathbb{C}^d}$ is a positive map, where $1_{\mathbb{C}^d}$ is the identity map on $\mathcal{B}(\mathbb{C}^d)$. Then, the following statement, which is called the multiplicativity of p-norm, was conjectured in [1] but was disproved later;

$$\nu_p(\Phi \otimes \Omega) = \nu_p(\Phi)\nu_p(\Omega) \tag{1.3}$$

for any $p \in (1, \infty]$ and for all quantum channels Φ and Ω. Note that the bound $\nu_p(\Phi \otimes \Omega) \geq \nu_p(\Phi)\nu_p(\Omega)$ is straightforward.

The first counterexample, which is called Werner-Holevo channel, was found in [17] for $p > 4.79$ and $m = n = 3$. Then later, the above conjecture was shown to be false for any $p > 1$ if we choose large enough m and n (the dimension of the input and output spaces) [5]. However when $p = 2$, for example, we still don't know whether or not there is a counterexample for (1.3) of low dimension. In this paper, we show, in Theorem 7 and Theorem 9, that for any Positive Trace-Preserving (PTP) map $\Phi : \mathcal{B}(\mathbb{C}^3) \to \mathcal{B}(\mathbb{C}^2)$ and any CP map $\Omega : \mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^n)$

$$\nu_p(\Phi \otimes \Omega) = \nu_p(\Phi)\nu_p(\Omega) \tag{1.4}$$

for $p = 2$ and $p \geq 4$ as a slight generalization of the corresponding result in [9]. This result is interesting as the Werner-Holevo channel is a map from $\mathcal{B}(\mathbb{C}^3)$ to $\mathcal{B}(\mathbb{C}^3)$ violating multiplicativity for $p > 4.79$. There are some general results in [3],[11],[13], where sufficient conditions for the multiplicativity were derived. However these sufficient conditions have not been verified in general.

The above conjecture attracted attention in the relation to the additivity conjecture [12]. The additivity conjecture was proven to be globally equivalent to the additivity of Holevo capacity and the additivity of entanglement of formation [15], however, it was disproved recently [4]. Although, the additivity does not hold in general it is still interesting to look for classes of channels for which the additivity is true. For this the multiplicativity for p close to 1 can be used to prove the additivity [1]. Under some conditions, the multiplicativity for rather large p implies the additivity [19].

2 Maps to $\mathcal{B}(\mathbb{C}^2)$

Suppose that ρ is a Hermitian operator of unit trace on $\mathbb{C}^2$. Then, there exists $\mathbf{w} \in \mathbf{R^3}$ such that

$$\rho = \bar{I} + \frac{1}{2}\sum_{k=1}^{3} w_k \sigma_k. \tag{2.1}$$

Here, $\bar{I} = I/2$ is the normalized identity and σ_k are the Pauli matrices. Note that ρ is positive semidefinite if and only if $\|\rho\|_2 = \|\mathbf{w}\|_2 \leq 1$, and ρ is a rank-one projection if and only if $\|\rho\|_2 = \|\mathbf{w}\|_2 = 1$. We identify a quantum state with a vector in the unit ball in $\mathbb{R}^3$. In this case, a pure state, which is a rank-one projection, corresponds to a point on the unit sphere. This unit ball is called the Bloch ball, denoted by B_1. Note that the center corresponds to the maximally mixed state. The following estimate is also important.

$$\|\rho\|_2 = \sqrt{\frac{1}{2} + \frac{1}{2}\sum_{k=1}^{3} w_k^2}. \tag{2.2}$$

Note that the 2-norm is determined by the distance from the center and then this fact shows that $\nu_2(\Phi)$ is also determined by the minimum radius of ball which includes $\Phi(B_1)$ the image of

the Bloch ball by Φ. This observation can be extended to $p \in (1, \infty]$ by using the majorization of eigenvalues.

The depolarizing channel on $\mathcal{B}(\mathbb{C}^d)$ is defined as

$$\Psi_\lambda(\rho) = \lambda\rho + (1-\lambda)\mathrm{tr}[\rho]\bar{I}. \tag{2.3}$$

Here, $\bar{I} = I/d$ and $0 \leq \lambda \leq 1$. Then, when $d = 2$ it acts on the above quantum states as follows.

$$\Psi_\lambda(\rho) = \bar{I} + \frac{1}{2}\sum_{k=1}^{3} \lambda\, w_k \sigma_k. \tag{2.4}$$

The depolarizing channel Ψ_λ compresses B_1 to the ball with radius λ, which is denoted by B_λ.

Theorem 1 *Any PTP map $\Phi : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^2)$ can be written in the form [1] of*

$$\Phi = \Psi_\lambda \circ M. \tag{2.5}$$

Here, Ψ_λ is the depolarizing channel on $\mathcal{B}(\mathbb{C}^2)$ and $M : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^2)$ is a PTP map which has a rank-one-projection output, so that

$$\nu_p(\Phi) = \nu_p(\Psi_\lambda) \quad p \in (1, \infty]. \tag{2.6}$$

Proof. First, recall that the depolarizing channel on $\mathcal{B}(\mathbb{C}^2)$ is defined by the following mappings.

$$\begin{gathered} \Psi_\lambda : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^2) \\ I \mapsto I; \qquad \sigma_1 \mapsto \lambda\sigma_1; \qquad \sigma_2 \mapsto \lambda\sigma_2; \qquad \sigma_3 \mapsto \lambda\sigma_3. \end{gathered} \tag{2.7}$$

We define a new map for $0 < \lambda \leq 1$:

$$\begin{gathered} L_\lambda : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^2) \\ I \mapsto I; \qquad \sigma_1 \mapsto \tfrac{1}{\lambda}\sigma_1; \qquad \sigma_2 \mapsto \tfrac{1}{\lambda}\sigma_2; \qquad \sigma_3 \mapsto \tfrac{1}{\lambda}\sigma_3. \end{gathered} \tag{2.8}$$

Then, next, choose $0 \leq \lambda \leq 1$ such that $\nu_p(\Phi) = \nu_p(\Psi_\lambda)$. Since when $\lambda = 0$ (Φ has only one output $\bar{I}$ and $\nu_2(\Phi) = 1/\sqrt{2}$) the statement of theorem holds, we assume that $\lambda > 0$. Then L_λ is well-defined and the channel Φ can be written as

$$\Phi = \Psi_\lambda \circ L_\lambda \circ \Phi. \tag{2.9}$$

Here, $\Psi_\lambda \circ L_\lambda$ acts as the identity.

Finally, we show the map $M = L_\lambda \circ \Phi$ is PTP and has a rank-one-projection output. Note that a TP map M is positive iff $M(B_1) \subseteq B_1$. The condition $\nu_p(\Phi) = \nu_p(\Psi_\lambda)$ implies that $\Phi(B_1)$ is touching B_λ from the inside. Hence,

$$M(B_1) = L_\lambda(\Phi(B_1)) \subseteq L_\lambda(B_\lambda) = B_1. \tag{2.10}$$

This shows that the map M is positive and that $M(B_1)$ is touching B_1 from inside so that M has a rank-one-projection output. By the construction M preserves trace.
Q.E.D.

Also, the following result on the depolarizing channels is well-known [7],[8].

[1] This form of decomposition may be traced back to our previous paper [2].

Theorem 2 *Let Ψ_λ be the depolarizing channel. Then, $\nu_p(\Psi_\lambda \otimes \Omega) \leq \nu_p(\Psi_\lambda)\, \nu_p(\Omega)$*

for any CP map Ω and $p \in (1, \infty]$.

3 Decomposability and its application

In this section, we use the concept of decomposability to prove multiplicativity properties for PTP maps between low dimensional spaces.

Definition 3 *A positive map M is decomposable if*

$$M = \Phi_1 + T \circ \Phi_2 \tag{3.1}$$

for some CP maps Φ_1 and Φ_2. Here, T is the transpose map.

The following result is well-known [16],[18] and our result totally depends on it.

Theorem 4 *All positive maps $M : \mathcal{B}(\mathbb{C}^3) \to \mathcal{B}(\mathbb{C}^2)$ and $M : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^3)$ are decomposable.*

Then, we have

Lemma 5 *Let Φ be a PTP map from $\mathcal{B}(\mathbb{C}^3)$ to $\mathcal{B}(\mathbb{C}^2)$. Then,*

$$\Phi = \Psi_\lambda \circ \Phi_1 + T \circ \Psi_\lambda \circ \Phi_2 \tag{3.2}$$

for some CP maps Φ_1 and Φ_2, so that $\nu_p(\Phi) = \nu_p(\Psi_\lambda)$ for $p \in (1, \infty]$.

Proof. By Theorem 1 and Theorem 4

$$\begin{aligned} \Phi &= \Psi_\lambda \circ M = \Psi_\lambda \circ [\Phi_1 + T \circ \Phi_2] \\ &= \Psi_\lambda \circ \Phi_1 + \Psi_\lambda \circ T \circ \Phi_2 = \Psi_\lambda \circ \Phi_1 + T \circ \Psi_\lambda \circ \Phi_2. \end{aligned} \tag{3.3}$$

Note that Ψ_λ and T are commutative.
Q.E.D.

3.1 For $p = 2$

When $p = 2$ we have the following nice property on the 2-norm:

Lemma 6

$$\|\hat{A}\|_2 = \|(T \otimes \mathbf{1}_{\mathbb{C}^n})(\hat{A})\|_2 \tag{3.4}$$

for any $\hat{A} \in \mathcal{B}(\mathbb{C}^{mn})$.

Proof. $\hat{A} \in \mathcal{B}(\mathbb{C}^{mn})$ can be written as

$$\hat{A} = \sum_{i,j=1}^{m} |i\rangle\langle j| \otimes A_{ij} \tag{3.5}$$

Here, $\{|i\rangle\}$ is an orthonormal basis and $A_{ij} \in \mathcal{B}(\mathbb{C}^n)$. Then,

$$(T \otimes \mathbf{1}_{\mathbb{C}^n})(\hat{A}) = \sum_{i,j=1}^{m} |j\rangle\langle i| \otimes A_{ij}. \tag{3.6}$$

Here, the transpose T is defined in the basis $\{|i\rangle\}$. Therefore,

$$\|\hat{A}\|_2^2 = \sum_{i,j=1}^{m} \|A_{ij}\|_2^2 = \|(T \otimes \mathbf{1}_{\mathbb{C}^n})(\hat{A})\|_2^2. \tag{3.7}$$

Q.E.D.

Theorem 7 *Let Φ be a PTP map from $\mathcal{B}(\mathbb{C}^3)$ to $\mathcal{B}(\mathbb{C}^2)$. Then, for any CP map $\Omega : \mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^n)$,*

$$\nu_2(\Phi \otimes \Omega) = \nu_2(\Phi)\, \nu_2(\Omega). \tag{3.8}$$

Proof. We show $\nu_2(\Phi \otimes \Omega) \leq \nu_2(\Phi)\, \nu_2(\Omega)$ as the other inequality is obvious.

For any state $\hat{\rho} \in \mathcal{D}(\mathbb{C}^3 \otimes \mathbb{C}^m)$ let σ_1 and σ_2 be positive semidefinite Hermitian operators as follows;

$$\sigma_1 = (\Phi_1 \otimes \mathbf{1})(\hat{\rho}) \text{ and } \sigma_2 = (\Phi_2 \otimes \mathbf{1})(\hat{\rho}). \tag{3.9}$$

Here, Φ_1 and Φ_2 are as in Lemma 5. Then,

$$(\Phi \otimes \mathbf{1})(\hat{\rho}) = (\Psi_\lambda \otimes \mathbf{1})(\sigma_1) + ((T \circ \Psi_\lambda) \otimes \mathbf{1})(\sigma_2) \tag{3.10}$$

Also, since Φ, Ψ_λ and T preserve trace,

$$1 = \mathrm{tr}[(\Phi \otimes \mathbf{1})(\hat{\rho})] = \mathrm{tr}[\sigma_1] + \mathrm{tr}[\sigma_2]. \tag{3.11}$$

Next, Theorem 2 gives the following bounds.

$$\|(\Psi_\lambda \otimes \Omega)(\sigma_1)\|_2 \leq \nu_2(\Psi_\lambda)\, \nu_2(\Omega)\, \mathrm{tr}[\sigma_1] \quad \text{and} \quad \|(\Psi_\lambda \otimes \Omega)(\sigma_2)\|_2 \leq \nu_2(\Psi_\lambda)\, \nu_2(\Omega)\, \mathrm{tr}[\sigma_2] \tag{3.12}$$

Then, by using (3.10), the triangle inequality, Lemma 6, (3.12) and (3.11) in order,

$$\begin{aligned} \|(\Phi \otimes \Omega)(\hat{\rho})\|_2 &\leq \|(\Psi_\lambda \otimes \Omega)(\sigma_1)\|_2 + \|(T \otimes \mathbf{1}) \circ (\Psi_\lambda \otimes \Omega)(\sigma_2)\|_2 \\ &= \|(\Psi_\lambda \otimes \Omega)(\sigma_1)\|_2 + \|(\Psi_\lambda \otimes \Omega)(\sigma_2)\|_2 \\ &\leq \nu_2(\Psi_\lambda)\, \nu_2(\Omega)\, [\mathrm{tr}[\sigma_1] + \mathrm{tr}[\sigma_2]] \\ &= \nu_2(\Phi)\, \nu_2(\Omega). \end{aligned} \tag{3.13}$$

This implies that

$$\nu_2(\Phi \otimes \Omega) \leq \nu_2(\Phi)\, \nu_2(\Omega). \tag{3.14}$$

Q.E.D.

3.2 For $p \geq 4$

To get the result for $p \geq 4$ we need the following result [9]. Note that it is also possible to use Theorem 8 instead of Lemma 6 to prove Theorem 7.

Theorem 8 *Let $A, B, C, D \in \mathcal{B}(\mathbb{C}^d)$ for $d \geq 1$. Then,*

$$\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\|_p \leq \left\| \begin{pmatrix} \|A\|_p & \|B\|_p \\ \|C\|_p & \|D\|_p \end{pmatrix} \right\|_p \tag{3.15}$$

for $p = 2$ and $p \geq 4$.

Theorem 9 *Let Φ be a PTP map from $\mathcal{B}(\mathbb{C}^3)$ to $\mathcal{B}(\mathbb{C}^2)$. Then, for any CP map $\Omega : \mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^n)$,*

$$\nu_p(\Phi \otimes \Omega) = \nu_p(\Phi)\, \nu_p(\Omega). \tag{3.16}$$

for $p \geq 4$.

Proof. We can prove the above statement in a similar way as Theorem 7. One step which is not trivial is the following bound:

$$\|(T \otimes \mathbf{1}) \circ (\Psi_\lambda \otimes \Omega)(\sigma_2)\|_p \leq \nu_p(\Psi_\lambda)\, \nu_p(\Omega)\, \mathrm{tr}[\sigma_2]. \tag{3.17}$$

Here, we use the same notations as in the proof of Theorem 7. To get this bound write

$$\sigma_2 = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \tag{3.18}$$

for some $A, B, C \in \mathcal{B}(\mathbb{C}^m)$. Note that since σ_2 is positive semidefinite, so are A and C. Then,

$$\|(T \otimes \mathbf{1}) \circ (\Psi_\lambda \otimes \Omega)(\sigma_2)\|_p = \left\| \begin{pmatrix} \frac{1+\lambda}{2}\Omega(A) + \frac{1-\lambda}{2}\Omega(C) & \lambda\Omega(B^*) \\ \lambda\Omega(B) & \frac{1-\lambda}{2}\Omega(A) + \frac{1+\lambda}{2}\Omega(C) \end{pmatrix} \right\|_p \tag{3.19}$$

By Theorem 8 and the triangle inequality, it is bounded by

$$\begin{aligned} & \left\| \begin{pmatrix} \frac{1+\lambda}{2}\|\Omega(A)\|_p + \frac{1-\lambda}{2}\|\Omega(C)\|_p & \lambda\|\Omega(B^*)\|_p \\ \lambda\|\Omega(B)\|_p & \frac{1-\lambda}{2}\|\Omega(A)\|_p + \frac{1+\lambda}{2}\|\Omega(C)\|_p \end{pmatrix} \right\|_p \\ = & \left\| \Psi_\lambda \left(\begin{pmatrix} \|\Omega(A)\|_p & \|\Omega(B^*)\|_p \\ \|\Omega(B)\|_p & \|\Omega(C)\|_p \end{pmatrix} \right) \right\|_p \\ \leq & \ \nu_p(\Psi_\lambda)\ [\|\Omega(A)\|_p + \|\Omega(C)\|_p] \\ \leq & \ \nu_p(\Psi_\lambda)\, \nu_p(\Omega)\, [\mathrm{tr}[A] + \mathrm{tr}[C]] \\ = & \ \nu_p(\Psi_\lambda)\, \nu_p(\Omega)\, \mathrm{tr}[\sigma_2]. \end{aligned} \tag{3.20}$$

Here, we used the fact that the following 2×2 matrices

$$\begin{pmatrix} \|\frac{1+\lambda}{2}\Omega(A) + \frac{1-\lambda}{2}\Omega(C)\|_p & \|\lambda\Omega(B^*)\|_p \\ \|\lambda\Omega(B)\|_p & \|\frac{1-\lambda}{2}\Omega(A) + \frac{1+\lambda}{2}\Omega(C)\|_p \end{pmatrix} \text{ and } \begin{pmatrix} \|\Omega(A)\|_p & \|\Omega(B^*)\|_p \\ \|\Omega(B)\|_p & \|\Omega(C)\|_p \end{pmatrix} \tag{3.21}$$

are positive semidefinite. Indeed, since

$$\begin{pmatrix} \Omega(A) & \Omega(B) \\ \Omega(B^*) & \Omega(C) \end{pmatrix} \tag{3.22}$$

is positive semidefinite we can write $\Omega(B) = \Omega(A)^{1/2} \, R \, \Omega(C)^{1/2}$ for some contraction R but this gives the bound: $\|\Omega(B)\|_p \le \sqrt{\|\Omega(A)\|_p \|\Omega(C)\|_p}$ and hence the positivity in (3.21).

Since the following bound:

$$\|(\Psi_\lambda \otimes \Omega)(\sigma_1)\|_p \le \nu_p(\Psi_\lambda)\, \nu_p(\Omega)\, \mathrm{tr}[\sigma_1] \tag{3.23}$$

is derived in a similar way we have

$$\|(\Phi \otimes \Omega)(\hat{\rho})\|_p \le \nu_p(\Phi)\, \nu_p(\Omega). \tag{3.24}$$

Q.E.D.

Remark 10 We take Ω as a CP map but the 2-positivity is sufficient. A similar observation holds in the following section as well.

3.3 Generalization and corollaries

Any CP map Φ from $\mathcal{B}(\mathbb{C}^m)$ to $\mathcal{B}(\mathbb{C}^n)$ can be written in the Kraus form:

$$\Phi(\rho) = \sum_{k=1}^{N} A_k \rho A_k^*. \tag{3.25}$$

Here, A_k are $n \times m$ matrices. The condition $\sum_{k=1}^N A_k^* A_k = I$ implies that Φ is TP. We also define the complementary/conjugate channel of Φ as follows.

$$\Phi^C(\rho) = \mathrm{tr}[A_k \rho A_l^*] |k\rangle\langle l|. \tag{3.26}$$

Note that this is a CPTP map from $\mathcal{B}(\mathbb{C}^m)$ to $\mathcal{B}(\mathbb{C}^N)$, whose dimension is the number of Kraus operators in (3.25). As in [6], [10], a channel and its complementary/conjugate channel share the maximal output p-norm and then the multiplicativity property. Therefore, Theorem 7 and Theorem 9 give the following corollary.

Corollary 11 *Let Φ be a CPTP map from $\mathcal{B}(\mathbb{C}^3)$ to $\mathcal{B}(\mathbb{C}^n)$. If Φ can be written by two Kraus operators then $\nu_p(\Phi \otimes \Omega) = \nu_p(\Phi)\, \nu_p(\Omega)$ for $p = 2$ and $p \ge 4$.*

Proof. Φ^C is a CPTP map from $\mathcal{B}(\mathbb{C}^3)$ to $\mathcal{B}(\mathbb{C}^2)$. Hence, by using Theorem 7 and Theorem 9, the statement follows.
Q.E.D.

Also, we can generalize Theorem 7:

Theorem 12 *Suppose we have a PTP map $\Phi = \Psi_\lambda \circ M$. Here, M is a PTP decomposable map from $\mathcal{B}(\mathbb{C}^m)$ to $\mathcal{B}(\mathbb{C}^n)$ having a rank-one-projection output, and Ψ_λ is the depolarizing channel on $\mathcal{B}(\mathbb{C}^n)$. Then $\nu_2(\Phi \otimes \Omega) = \nu_2(\Phi)\, \nu_2(\Omega)$ for any CP map Ω.*

The above statement can be proven in a similar way as Theorem 7, and it is a generalization of the result in [2] when $p = 2$. Note that this statement is not vacuous. For example, take two CPTP maps Φ_1 and Φ_2 such that Φ_1 and $T \circ \Phi_2$ have the common rank-one-projection output. Then, $M = q\,\Phi_1 + (1-q)\,T \circ \Phi_2$ for $0 \le q \le 1$ satisfies the above condition.

Corollary 13 *Suppose we have a PTP map $\Phi = \Psi_\lambda \circ M$. Here, M is a PTP map from $\mathcal{B}(\mathbb{C}^2)$ to $\mathcal{B}(\mathbb{C}^3)$ having a rank-one-projection output, and Ψ_λ is the depolarizing channel on $\mathcal{B}(\mathbb{C}^3)$. Then $\nu_2(\Phi \otimes \Omega) = \nu_2(\Phi)\,\nu_2(\Omega)$ for any CP map Ω.*

Proof. By Theorem 4, M is always decomposable. Hence by Theorem 12 the result follows. Q.E.D.

4 Discussion

In this paper, we used the concept of decomposability of positive maps. Since partial transpose does not preserve positivity we had to exclude the case $p \in (2, 4)$. It would be interesting to investigate whether or not the same bound holds for $p \in (2, 4)$. There is another interesting question. We don't know very much about decomposability of positive maps $M : \mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^2)$ when $m > 3$ although some researches are being done [14]. Decomposable maps of this class will give other PTP maps which have multiplicativity property.

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